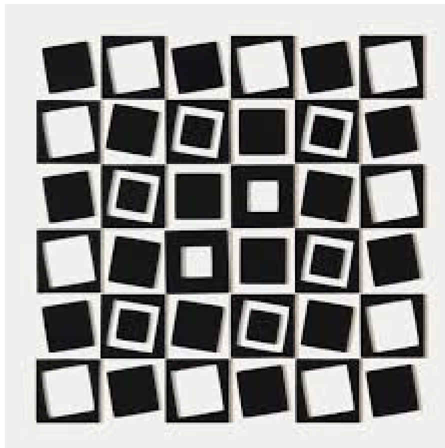
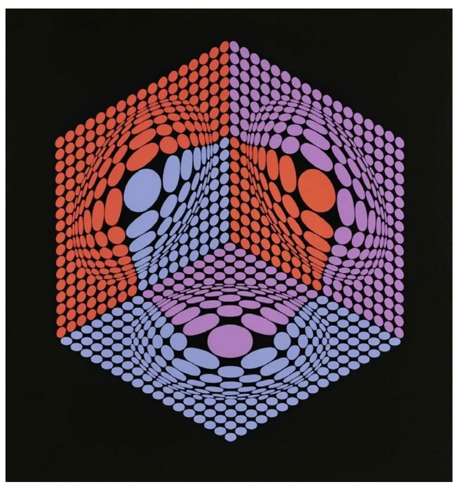
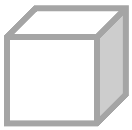
VIII.3 Method of Separation of Variables – Transient Initial-Boundary Value Problems





THE HEAT EQUATION 3-D Cartesian Coordinates

$u(x,y,z,t)$



$[u]_S = f$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+F(x,y,z) = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad (x,y,z)\in(0,L)\times(0,M)\times(0,K)\subset\mathbb{R}^3,\ t>0$$

$$[u]_S = f \qquad t>0$$

$$[u]_{t=0} = u_0(x,y,z) \qquad t=0 \qquad (x,y,z)\in[0,L]\times[0,M]\times[0,K]\subset\mathbb{R}^3$$

superposition

$$u(x,y,z,t) = u_{ss}(x,y,z) + U(x,y,z,t)$$

$u_{ss}(x,y,z)$ $U(x,y,z,t)$

STEADY STATE PROBLEM - PELE

$$\frac{\partial^2 u_{ss}}{\partial x^2}+\frac{\partial^2 u_{ss}}{\partial y^2}+\frac{\partial^2 u_{ss}}{\partial z^2} + F(x,y,z)=0$$

Laplace Eqn $\nabla^2 u = 0$ $[u_{ss}]_S = f$ → *six basic problems*: 0 u_k f_k

Poisson Eqn $\nabla^2 u + F = 0$: 0 u_6 0

TRANSIENT PROBLEM (basic)

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$[U]_S = 0 \qquad [U]_{t=t_0} = u_0 - u_{ss}$$

supplemental eigenvalue problems

$$X'' = \mu X,\quad [X]_{x=0}=0,\quad [X]_{x=L}=0 \quad \overset{SLP}{\Rightarrow} \quad X_n'' = -\lambda_n^2 X_n,\quad \mu_n = -\lambda_n^2,\quad X_n(x)$$

$$Y'' = \eta Y,\quad [Y]_{y=0}=0,\quad [Y]_{y=M}=0 \quad \overset{SLP}{\Rightarrow} \quad Y_m'' = -\nu_m^2 Y_m,\quad \eta_m = -\nu_m^2,\quad Y_m(y)$$

$$Z'' = \gamma Z,\quad [Z]_{z=0}=0,\quad [Z]_{z=K}=0 \quad \overset{SLP}{\Rightarrow} \quad Z_k'' = -\omega_k^2 Z_k,\quad \gamma_k = -\omega_k^2,\quad Z_k(z)$$

SEPARATION OF VARIABLES

$$U(x,y,z,t) = \Phi(x,y,z)\,T(t) \qquad \frac{\nabla^2\Phi}{\Phi} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

HELMHOLTZ EQUATION

$$\nabla^2\Phi = \beta\Phi$$

$$\Phi(x,y,z) = X(x)Y(y)Z(z)$$

$$\lambda_n,\quad \nu_m,\quad \omega_k$$

$$X_n,\quad Y_m,\quad Z_k$$

$$\beta_{nmk} = -\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)$$

$$\frac{1}{\alpha}\frac{T'}{T} = \beta \qquad T = e^{-\alpha\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)t}$$

STEADY STATE SOLUTION (PELE)

$$u_{ss}(x,y,z) = u_1 + u_2 + \dots + u_6 + \sum_n\sum_m\sum_k A_{nmk} X_n Y_m Z_k$$

$$\text{where} \quad A_{mnk} = \frac{-\int_0^K\int_0^M\int_0^L F(x,y,z)X_n Y_m Z_k\,dx\,dy\,dz}{\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)\|X_n\|^2\,\|Y_m\|^2\,\|Z_k\|^2}$$

$u_{ss}(x,y,z)$ = *solution of six basic problems for Laplace's Equation plus solution of Poisson's equation with zero b.c.'s*

TRANSIENT SOLUTION

$$U(x,y,z,t) = \sum_n\sum_m\sum_k B_{nmk}\, X_n Y_m Z_k\, e^{-\alpha\left(\lambda_n^2+\nu_m^2+\omega_k^2\right)t}$$

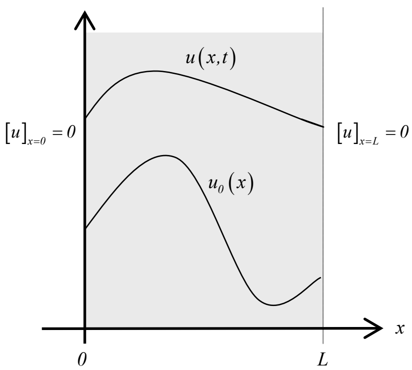
$$\text{where} \quad B_{nmk} = \frac{\int_0^K\int_0^M\int_0^L (u_0-u_s)X_n Y_m Z_k\,dx\,dy\,dz}{\|X_n\|^2\,\|Y_m\|^2\,\|Z_k\|^2}$$

THE SOLUTION OF THE IBVP is a superposition of the steady-state and transient solutions

$$u(x,y,z,t) = u_{ss}(x,y,z) + U(x,y,z,t)$$

VIII.3.1 HEAT EQUATION IN PLANE WALL – 1-D Heat Equation BASIC

VIII.3.1.1 BASIC CASE: Homogeneous equation, Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad x\in(0,L)\,,\ t>0$$

Initial condition: $u(x,0)=u_0(x)$

homogeneous b.c.

Boundary conditions: $[u]_{x=0}=0\,,\ t>0$ (*I, II or III kind*)

$[u]_{x=L}=0\,,\ t>0$ (*I, II or III kind*)

1) Separation of variables: $u(x,t)=X(x)T(t)$

Boundary conditions:

$$[u]_{x=0}=[X]_{x=0}\,T(t)=0 \quad\Rightarrow\quad [X]_{x=0}=0$$
$$[u]_{x=L}=[X]_{x=L}\,T(t)=0 \quad\Rightarrow\quad [X]_{x=L}=0$$

$$\frac{X''}{X}=\frac{1}{\alpha}\frac{T'}{T}=\mu$$

2) Sturm-Liouville Problem:

$$X''-\mu X=0$$
$$[X]_{x=0}=0 \quad\Rightarrow\quad \mu=-\lambda_n^2 \qquad n=1,2,\ldots$$
$$[X]_{x=L}=0 \qquad\qquad X_n(x)$$

3) Equation for T:

$$T'-\alpha\mu T=0$$

$$T'+\alpha\lambda_n^2 T=0 \quad\Rightarrow\quad T_n(t)=e^{-\alpha\lambda_n^2 t}$$

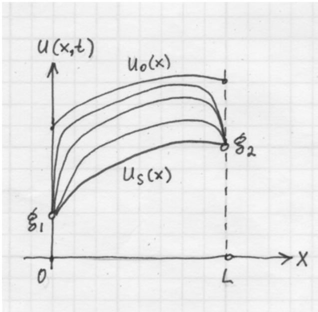
4) Solution:

$$u(x,t)=\sum_{n=1}^{\infty}a_n X_n T_n \qquad =\sum_{n=1}^{\infty} a_n\, X_n\ e^{-\alpha\,\lambda_n^2\,t}$$

Initial condition:

$$u(x,0)=u_0(x)=\sum_{n=1}^{\infty}a_n X_n \quad\Rightarrow\quad a_n=\frac{\int_0^L u_0(x)X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

Example 1 ***Dirichlet-Dirichlet problem with a uniform heat generation***



$$\frac{\partial^2 u}{\partial x^2}+F=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t):\ x\in(0,L),\quad t>0$$

Initial condition: $u(x,0)=u_0(x)$

Boundary conditions: $u(0,t)=g_1 \quad t>0 \quad (Dirichlet)$

$u(L,t)=g_2 \quad t>0 \quad (Dirichlet)$

1) Steady State Solution:

Let $F=const$, then integrating the equation twice, we come up with the following solution:

$$\frac{\partial u_s}{\partial x}=-F\,x+c_1$$

$$u_s=-\frac{F}{2}x^2+c_1x+c_2$$

Apply boundary conditions to determine the constants of integration:

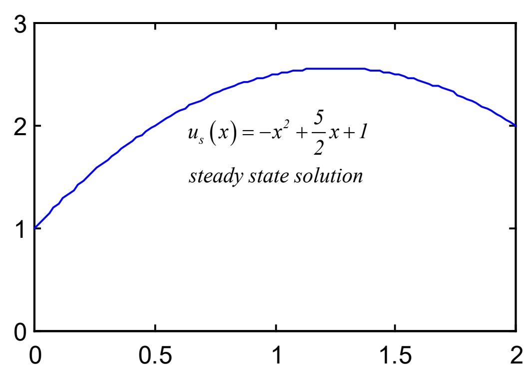
$\underline{x=0} \quad \Rightarrow \quad c_2=g_1$

$\underline{x=L} \quad \Rightarrow \quad -\frac{F}{2}L^2+c_1L+g_1=g_2$

$\Rightarrow \quad c_1=\frac{g_2-g_1}{L}+\frac{FL}{2}$

$$\boxed{u_s(x)=-\frac{F}{2}x^2+\left(\frac{g_2-g_1}{L}+\frac{FL}{2}\right)x+g_1}$$

Example: $F=2,\ g_1=1,\ g_2=2,\ L=2$



2) Transient Solution:

$$\frac{\partial^2 U}{\partial x^2}=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U(x,t):\ x\in(0,L),\ t>0$$

initial condition: $U(x,0)=u_0(x)-u_s(x)$

boundary conditions: $U(0,t)=0 \quad$ (Dirichlet)

$U(L,t)=0 \quad$ (Dirichlet)

Solution of this basic problem (Dirichlet-Dirichlet) obtained by separation of variables:

$$\lambda_n = \frac{n\pi}{L}, \qquad X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

$$U(x,t) = \sum_{n=1}^{\infty} a_n X_n e^{-\alpha\lambda_n^2 t} = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{\alpha n^2\pi^2}{L^2}t}$$

where coefficients a_n are the Fourier coefficients determined by the corresponding initial condition for the function $U(x,t)$:

$$a_n = \frac{\int_0^L \left[u_0(x) - u_s(x)\right] X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx} = \frac{2}{L}\int_0^L \left[u_0(x) - u_s(x)\right] \sin\left(\frac{n\pi}{L}x\right) dx$$

3) Solution of IBVP:
Return to the original function $u(x,t)$:

$$u(x,t) = U(x,t) + u_s(x) = u_s(x) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{\alpha n^2\pi^2}{L^2}t}$$

Then the solution of the non-homogeneous heat equation with non-homogeneous Dirichlet boundary conditions becomes:

$$u(x,t) = \left[-\frac{F}{2}x^2 + \left(\frac{g_2 - g_1}{L} + \frac{FL}{2}\right)x + g_1\right] + \frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L \left[u_0(x) - u_s(x)\right]\sin\left(\frac{n\pi}{L}x\right)dx\right\}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\alpha n^2\pi^2}{L^2}t}$$

Remark: In practice, instead of the exact solution defined by the infinite series, the truncated series is used for calculation of the approximate solution. How many terms are needed in the truncated series for the accurate approximation? Comparison of the exact solution (which is also a truncated series but with a very large number of terms, which we assume, provides an accurate result) with the calculation with a small number of terms in a truncated series shows that the accuracy depends on time: the further we proceed in time, the more accurate becomes an approximate solution (why?). For uniform characterization of physical processes, the non-dimensional parameters are used in engineering. In heat transfer, non-dimensional time is defined by the Fourier number:

$$Fo = \frac{\alpha t}{L^2}$$

where α is the *thermal diffusivity*.

In engineering heat transfer analysis, a 4 term approximation is considered as an accurate approximation for all values of the Fourier number. For simplicity, very often even a 1 term approximation is used, which is considered to be accurate for $Fo > 0.2$ (error in most cases does not exceed *1%,* and this is a convention in engineering heat transfer).

Consider comparison of the exact solution (100 terms) with 1 and 4 terms approximations.

Results are calculated for:

$Fo = 0.0$

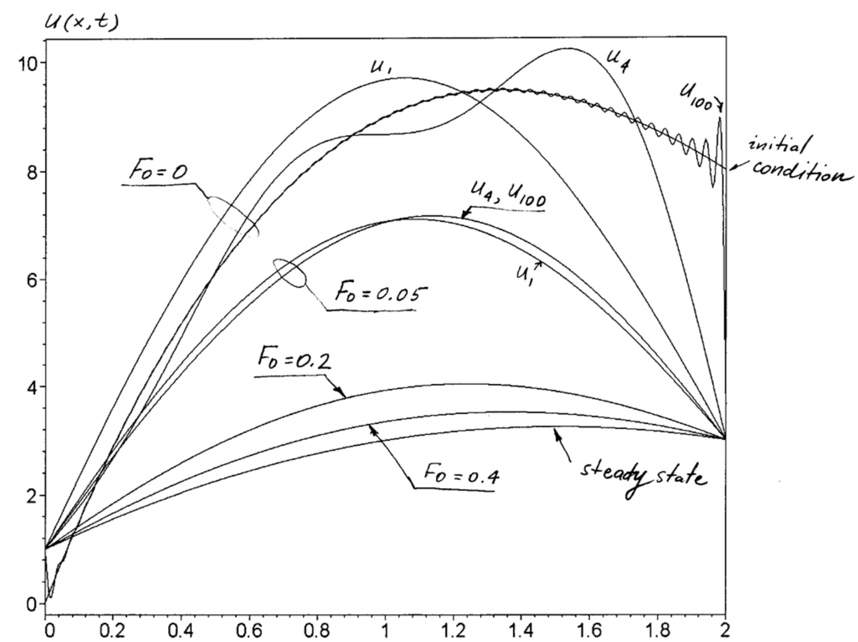
$Fo = 0.05$

$Fo = 0.2$

$Fo = 0.4$

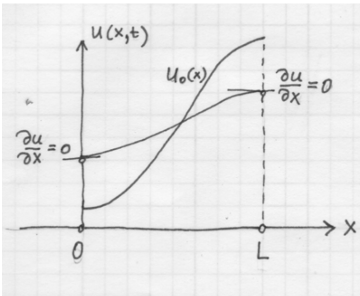
The lowest curve is a steady state solution.

As can be seen from the figure, for $Fo > 0.2$, all results coincide.



$Fo > 0.2$	is generally adopted as a condition for application of one term approximation: one-term solution becomes accurate for $Fo > 0.2$.

Example 2 ***Neumann-Neumann Problem (thermoinsulated walls)***



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in (0,L), \quad t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions:

$$\left[\frac{\partial u}{\partial x}\right]_{x=0} = 0 \qquad t > 0 \qquad (Neumann)$$

$$\left[\frac{\partial u}{\partial x}\right]_{x=L} = 0 \qquad t > 0 \qquad (Neumann)$$

(both boundaries are insulated)

Separation of variables: $u(x,t) = X(x)T(t)$

Boundary conditions:

$$x = 0 \qquad \frac{\partial u(0,t)}{\partial x} = X'(0)\,T(t) = 0 \qquad \Rightarrow \quad X'(0) = 0$$

$$x = L \qquad \frac{\partial u(L,t)}{\partial x} = X'(L)T(t) = 0 \qquad \Rightarrow \quad X'(L) = 0$$

Solution of SLP:

$$X'' - \mu X = 0 \qquad \mu_n = -\lambda_n^2$$

$$X'(0) = 0 \qquad \lambda_0 = 0 \qquad X_0 = 1$$

$$X'(L) = 0 \qquad \lambda_n = \frac{n\pi}{L} \qquad X_n = cos\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, ...$$

Solution for T:

$$T' + \alpha \cdot 0 \cdot T = 0 \qquad T_0(t) = 1$$

$$T' + \alpha\lambda_n^2 T = 0 \qquad T_n(t) = e^{-\alpha\lambda_n^2 t}$$

Solution:

$$u(x,t) = a_0 X_0 T_0 + \sum_{n=1}^{\infty} a_n X_n T_n \qquad = a_0 + \sum_{n=1}^{\infty} a_n X_n e^{-\alpha\lambda_n^2 t}$$

$$a_0 = \frac{1}{L}\int_0^L u_0(x)\,dx$$

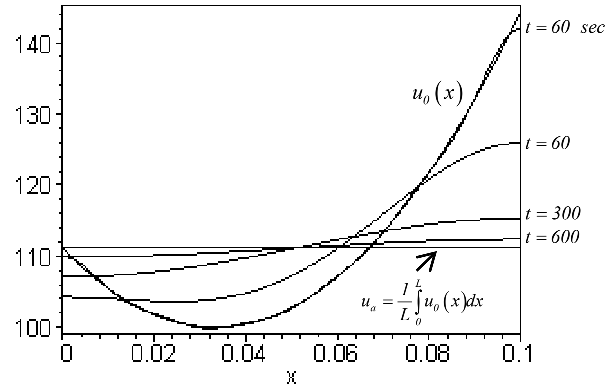
$$a_n = \frac{2}{L}\int_0^L u_0(x)\,cos\left(\frac{n\pi}{L}x\right)dx$$

Solution of IBVP:

$$u(x,t) = \frac{\int_0^L u_0(x)\,dx}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L u_0(x)\,cos\left(\frac{n\pi}{L}x\right)dx\right] cos\left(\frac{n\pi}{L}x\right) e^{-\alpha\frac{n^2\pi^2}{L^2}t}$$

Particular case:

$$u_0(x) = 100 + 10000\left(x - \frac{L}{3}\right)^2 \left[{}^oC\right], \quad \frac{1}{\alpha} = 500^2\left[\frac{s}{m^2}\right] (steel), \qquad L=0.1m$$



Comments:

1) The solution is in the form of an infinite series.
   If the initial temperature distribution given by the function $u_0(x)$ is integrable, then the Fourier series is absolutely convergent and the function $u(x,t)$ satisfies the Heat Equation and the initial and boundary conditions.

   Therefore, it is an *analytical solution* of the given IBVP.

2) With the increase of time, the solution approaches the steady state (the averaged temperature in the slab). Boundaries are insulated, and there are no heat sources. As a result, no heat escapes into the surroundings. The driving force – temperature gradient – is directed toward the areas with lower temperature. There exists a process of redistribution of heat energy that produces the uniform temperature in the slab.

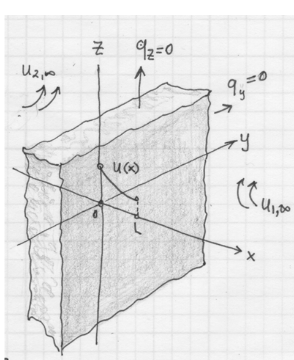
3) Basic functions consist of the product

$$u_n(x,t) = cos\left(\frac{n\pi}{L}x\right)e^{-\alpha\frac{n^2\pi^2}{L^2}t} = cos\left(\frac{n\pi}{L}x\right)e^{-(n\pi)^2 \overbrace{\left(\frac{\alpha t}{L^2}\right)}^{\substack{\text{non-dimensional time}\\ Fo}}}$$

   where the cosine function provides the spatial shape of the temperature profile; and the exponential function is responsible for decay of the temperature profile in time.

4) The rate of change of temperature depends on the thermal diffusivity α .

5) Very often, a *1-D* Heat Equation is treated as a model for heat transfer in a long very thin rod of constant cross-section whose surface, except for the ends, is insulated against the flow of heat Although, it is formally a correct model, the practical application of it is very limited. But there is another interpretation of a *1-D* model, which is more reliable.



   Consider a *3-D* wall with finite dimension in the x-direction (within $x = 0$ and $x = L$) and elongated dimensions (may be infinite) in y- and z-directions. If the conditions at the walls $x = 0$ and $x = L$ are uniform, and the initial condition is independent of variables *y* and *z*, then the variation of temperature in the *y*- and *z*-directions is negligible (no heat flux in these directions)

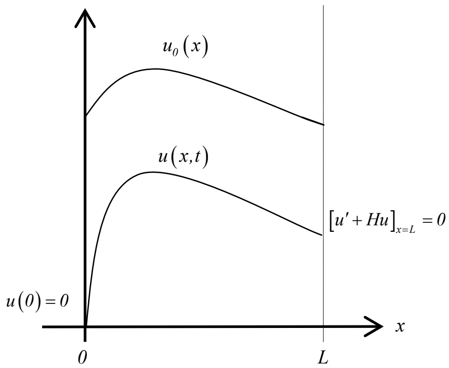
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$$

   and the heat equation becomes *1-D*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

It defines the variation of temperature along any line perpendicular to the wall.

Example 3 **Dirichlet-Robin Problem**



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad x \in (0,L), \quad t > 0$$

Initial condition: $u(x,0) = u_0(x)$

Boundary conditions: $[u]_{x=0} = 0$ (*Dirichlet*)

$$\left[\frac{\partial u}{dx} + Hu\right]_{x=L} = 0 \qquad (Robin) \qquad H = \frac{h}{k}$$

Separation of variables: $u(x,t) = X(x)T(t)$

$$X'' - \mu X = 0 \qquad T' - \alpha\mu T = 0$$

Boundary conditions :

$\underline{x = 0}$ $\quad X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$

$\underline{x = L}$ $\quad X'(L)T(t) + HX(L)T(t) = 0 \quad \Rightarrow \quad X'(L) + HX(L) = 0$

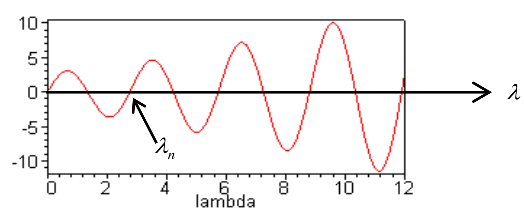
Solution of Sturm-Liouville problem:

$\mu_n = -\lambda_n^2$

$X_n = sin(\lambda_n x) \qquad n = 1,2,...$

where eigenvalues λ_n are positive roots of the characteristic equation:

$\lambda \cos \lambda L + H \sin \lambda x = 0$



Solution for *T(t):*

With determined eigenvalues, the solution for T becomes:
$T_n(t) = e^{-\alpha\lambda_n^2 t}$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \, sin(\lambda_n x) e^{-\alpha\lambda_n^2 t}$$

This solution satisfies the heat equation and boundary conditions. We want to define coefficients a_n in a such a way that the obtained solution satisfies also the initial condition at $t = 0$:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \, sin(\lambda_n x) = u_0(x)$$

In our problem, functions $\left\{X_n(x)=sin(\lambda_n x)\right\}$ are obtained as eigenfunctions of the Sturm-Liouville problem for the equation $X''+\lambda^2 X=0$; therefore, the set of all eigenfunctions is a complete system of functions orthogonal with respect to the weight function $p=1$. Then, the last equation is an expansion of the function $u_0(x)$ in a generalized Fourier series over the interval $(0,L)$ with coefficients defined by

$$a_n=\frac{\int_0^L u_0(x)\,sin(\lambda_n x)\,dx}{\int_0^L sin^2(\lambda_n x)\,dx}$$

Then, the solution of the initial-boundary value problem is given by

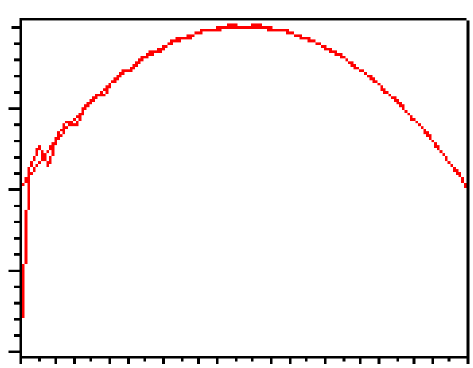
$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{\int_0^L u_0(x)\,sin(\lambda_n x)\,dx}{\int_0^L sin^2(\lambda_n x)\,dx}\right]sin(\lambda_n x)\,e^{-\alpha\lambda_n^2 t}$$

where the squared norm of eigenfunctions may be evaluated after integration as

$$\left\|X_n\right\|^2=\int_0^L sin^2(\lambda_n x)\,dx=\frac{L}{2}-\frac{sin(2\lambda_n L)}{4\lambda_n}$$

Finally, the solution is:

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{\int_0^L u_0(x)\,sin(\lambda_n x)\,dx}{\frac{L}{2}-\frac{sin(2\lambda_n L)}{4\lambda_n}}\right]sin(\lambda_n x)\,e^{-\alpha\lambda_n^2 t}$$



MAPLE: Let $L = 2$, $H = 3$, $u_0(x) = x(2-x)$, $\alpha = 0.0625$

```
> restart;
> with(plots):
> L:=2;H:=3;A:=0.0625;
```

$$L := 2$$

$$H := 3$$

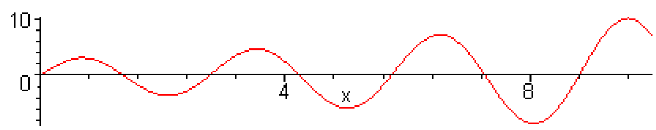
$$A := 0.0625$$

Characteristic equation:

```
> w(x):=x*cos(x*L)+H*sin(x*L);
```

$$w(x) := x\cos(2x) + 3\sin(2x)$$

```
> plot(w(x),x=0..10);
```



Eigenvalues:

```
> n:=1: for m from 1 to 500 do z:=fsolve(w(x)=0,x=m/10..(m+1)/10):
if type(z,float) then lambda[n]:=z: n:=n+1 fi od:
> for i to 5 do lambda[i] od;
```

$$1.358229874$$

$$2.768911636$$

$$4.235147453$$

$$5.738636645$$

$$7.264403196$$

```
> N:=n-1;
```

$$N := 32$$

```
> n:='n':i:='i':
```

Eigenfunctions:

```
> X[n]:=sin(lambda[n]*x);
```

$$X_n := \sin(\lambda_n x)$$

Squared-norm:

```
> NX[n]:=int(X[n]^2,x=0..L);
```

$$NX_n := \frac{1}{2}\,\frac{-\cos(2\lambda_n)\sin(2\lambda_n) + 2\lambda_n}{\lambda_n}$$

Initial condition:

```
> u0(x):=x*(L-x)+1;
```

$$u0(x) := x(2-x) + 1$$

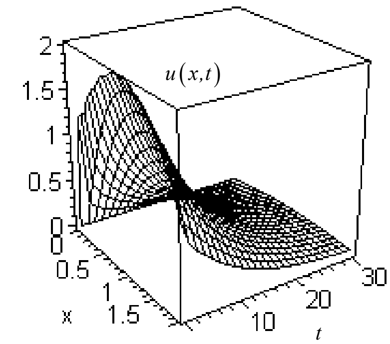
Fourier coefficients:

```
> a[n]:=simplify(int(u0(x)*X[n],x=0..L)/NX[n]);
```

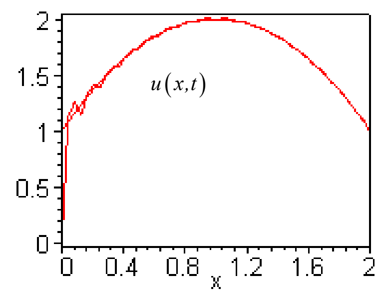
$$a_n := -\frac{2\left(2\lambda_n\sin(2\lambda_n) + \lambda_n^2\cos(2\lambda_n) + 2\cos(2\lambda_n) - 2 - \lambda_n^2\right)}{\lambda_n^2\left(-\cos(2\lambda_n)\sin(2\lambda_n) + 2\lambda_n\right)}$$

Solution - Generalized Fourier series:

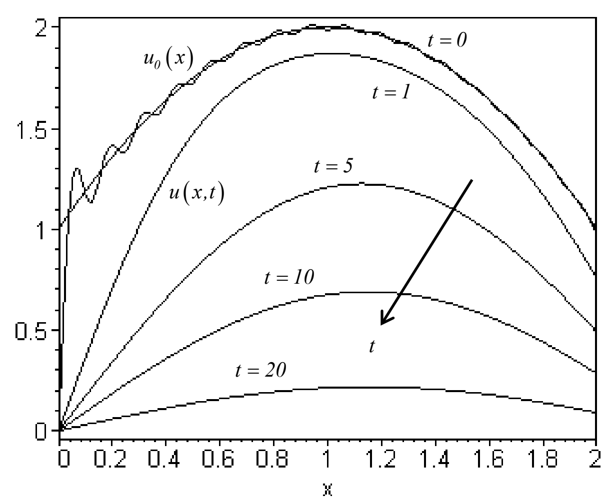
```
> u(x,t):=sum(a[n]*X[n]*exp(-lambda[n]^2*t/A^2),n=1..N):
> plot3d(u(x,t),x=0..L,t=0..30,axes=boxed,style=wireframe);
```



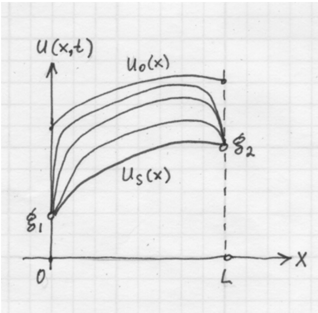

```
> animate({u0(x),u(x,t)},x=0..L,t=0..50,frames=200,axes=boxed);
```




```
> u(x,0):=subs(t=0,u(x,t)):
> u(x,1):=subs(t=1,u(x,t)):
> u(x,5):=subs(t=5,u(x,t)):
> u(x,10):=subs(t=10,u(x,t)):
> u(x,20):=subs(t=20,u(x,t)):
> plot({u0(x),u(x,0),u(x,1),u(x,5),u(x,10),u(x,20)},x=0..L);
```



VIII.3.1.2 GENERAL CASE Non-Homogeneous Equation, Non-Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t),\quad x\in(0,L),\quad t>0$$

Initial condition: $u(x,0)=u_0(x)$

Boundary conditions: $[u]_{x=0}=g_1,\ t>0$ (*I, II or IIIrd kind*)
$[u]_{x=L}=g_2,\ t>0$ (*I, II or IIIrd kind*)

I Steady State Solution

Definition A time-independent function which satisfies the heat equation and boundary conditions obtained as

$$u_s(x)=\lim_{t\to\infty}u(x,t)$$

is called a ***steady state solution***

Substitution of a time-independent function into the heat equation leads to the following ordinary differential equation:

$$\frac{\partial^2 u_s}{\partial x^2}+F(x)=0 \qquad u_s(x),\quad x\in(0,L)$$

subject to the boundary conditions of the same kind as for PDE:

$[u_s]_{x=0}=g_1,\ t>0$ (*I, II or IIIrd kind*)
$[u_s]_{x=L}=g_2,\ t>0$ (*I, II or IIIrd kind*)

General solution of ODE:

$$u_s(x)=-\int\left[\int F(x)\,dx\right]dx+c_1x+c_2$$

Solutions of BVPs for plane wall with uniform heat generation are provided by the Table.

II Transient Solution

Define the transient solution by equation:

$$U(x,t)=u(x,t)-u_s(x)$$

then solution of the original problem is a sum of transient solution and steady state solution:

$$u(x,t)=U(x,t)+u_s(x)$$

Substitute it into the Heat Equation:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 u_s}{\partial x^2}+F(x)=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

Since $\dfrac{\partial^2 u_s}{\partial x^2}+F(x)=0$, it yields

$$\frac{\partial^2 U}{\partial x^2}=a^2\frac{\partial U}{\partial t}$$

We obtained the equation for the new unknown function $U(x,t)$ which has homogeneous boundary conditions:

$$x=0 \qquad [U]_{x=0}=[u]_{x=0}-[u_s]_{x=0}=g_1-g_1=0$$

$$x=L \qquad [U]_{x=L}=[u]_{x=L}-[u_s]_{x=L}=g_2-g_2=0$$

As a result, we reduced the non-homogeneous problem to a homogeneous equation for $U(x,t)$ with homogeneous boundary conditions. Initial condition for function $U(x,t)$:

$$U(x,0)=u(x,0)-u_s(x)=u_0(x)-u_s(x)$$

Solution for *U(x,t)*

We consider the following ***basic*** initial boundary value problem:

$$\frac{\partial^2 U}{\partial x^2}=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U(x,t),\quad x\in(0,L),\ t>0$$

initial condition: $\quad U(x,0)=u_0(x)-u_s(x)$

boundary conditions: $\quad [U]_{x=0}=0,\quad t>0$

$\quad [U]_{x=L}=0 \quad t>0$

We already know a solution of this basic problem obtained by separation of variables:

$$U(x,t)=\sum_{n=1}^{\infty}a_n X_n e^{-\alpha\lambda_n^2 t}$$

where coefficients a_n are the Fourier coefficients determined with the corresponding initial condition for the function $U(x,t)$:

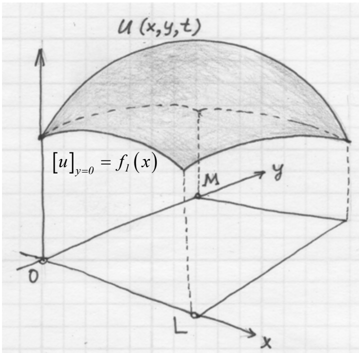
$$a_n=\frac{\int_0^L\left[u_0(x)-u_s(x)\right]X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}$$

<u>III Solution of IBVP</u>:

Solution of the original IBVP is a sum of steady state solution and transient solution:

$$\boxed{\begin{aligned} u(x,t) &= u_s(x)+U(x,t)\\ &= u_s(x)+\sum_{n=1}^{\infty}a_n X_n e^{-\alpha\lambda_n^2 t}\\ a_n &= \frac{\int_0^L\left[u_0(x)-u_s(x)\right]X_n(x)\,dx}{\int_0^L X_n^2(x)\,dx}\end{aligned}}$$

VIII.3.2.1 HEAT EQUATION in CARTESIAN COORDINATES 2-D General Problem



$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+F(x,y)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,y,t): \quad (x,y)\in(0,L)\times(0,M)$$

Initial Condition: $u(x,y,0)=u_0(x,y) \qquad (x,y)\in[0,L]\times[0,M]$

Boundary Conditions:

$$\begin{array}{llll} x=0 & [u]_{x=0}=f_3(y) & y\in(0,M) & t>0 \\ x=L & [u]_{x=L}=f_4(y) & y\in(0,M) & t>0 \\ y=0 & [u]_{y=0}=f_1(x) & x\in(0,L) & t>0 \\ y=M & [u]_{y=M}=f_2(x) & x\in(0,L) & t>0 \end{array}$$

1. Steady State Solution

Find time-independent solution $u_s(x,y)$. We are looking for a steady state solution which satisfies the differential equation:

$$\frac{\partial^2 u_s}{\partial x^2}+\frac{\partial^2 u_s}{\partial y^2}+F(x,y)=0$$

and the boundary conditions of the same type as in the general problem

$$\begin{array}{llll} x=0 & [u_s]_{x=0}=f_3(y) & y\in(0,M) & t>0 \\ x=L & [u_s]_{x=L}=f_4(y) & y\in(0,M) & t>0 \\ y=0 & [u_s]_{y=0}=f_1(x) & x\in(0,L) & t>0 \\ y=M & [u_s]_{y=M}=f_2(x) & x\in(0,L) & t>0 \end{array}$$

This is the BVP for Poisson's Equation for which, in general, all boundary conditions are non-homogeneous. The superposition principle should be used to reduce the problem to the set of supplemental basic problems (see VIII.3.4, p.597).

2. Transient Solution (Basic Case)

Introduce the transient function as

$$U(x,y,t)=u(x,y,t)-u_s(x,y)$$

It can be verified that function U satisfies *homogeneous* Heat Equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

with four *homogeneous* boundary conditions (of the same type):

$$\begin{array}{llll} x=0 & [U]_{x=0}=0 & y\in(0,M) & t>0 \\ x=L & [U]_{x=L}=0 & y\in(0,M) & t>0 \\ y=0 & [U]_{y=0}=0 & x\in(0,L) & t>0 \\ y=M & [U]_{y=M}=0 & x\in(0,L) & t>0 \end{array}$$

and the initial condition:

$$U(x,y,0)=u_0(x,y)-u_s(x,y) \quad \equiv U_0(x,y)$$

Separation of variables – 1st stage:

We assume that the function $U(x,y,t)$ can be written as a product of two functions

$$U(x,y,t)=\Phi(x,y)T(t)$$

where $\Phi(x,y)$ is the function of space variables. Substitute it into the Heat Equation

$$\frac{\partial^2\Phi}{\partial x^2}T+\frac{\partial^2\Phi}{\partial y^2}T=\frac{1}{\alpha}\Phi T'$$

Divide equation by ΦT:

$$\frac{\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2}}{\Phi}=\frac{1}{\alpha}\frac{T'}{T}$$

or using Laplacian operator

$$\frac{\nabla^2\Phi}{\Phi}=\frac{1}{\alpha}\frac{T'}{T}$$

Left hand side is a function of space variables only and the right hand side is a function of the time variable, therefore, they have to be equal to a constant (separation constant):

$$\boxed{\frac{\nabla^2\Phi}{\Phi}=\frac{1}{\alpha}\frac{T'}{T}=\beta}$$

Boundary conditions for separated functions are:

$$\left[U\right]_{x=0}=\left[\Phi\right]_{x=0}T(t)=0 \quad y\in(0,M) \quad t>0 \quad \Rightarrow \quad \left[\Phi\right]_{x=0}=0$$

$$\left[U\right]_{x=L}=\left[\Phi\right]_{x=L}T(t)=0 \quad y\in(0,M) \quad t>0 \quad \Rightarrow \quad \left[\Phi\right]_{x=L}=0$$

$$\left[U\right]_{y=0}=\left[\Phi\right]_{y=0}T(t)=0 \quad x\in(0,L) \quad t>0 \quad \Rightarrow \quad \left[\Phi\right]_{y=0}=0$$

$$\left[U\right]_{y=M}=\left[\Phi\right]_{y=M}T(t)=0 \quad x\in(0,L) \quad t>0 \quad \Rightarrow \quad \left[\Phi\right]_{y=M}=0$$

There are four homogeneous boundary conditions for the function Φ.

From the separated equations, consider the equation

Helmholtz Equation

$$\boxed{\nabla^2\Phi=\beta\Phi}$$

which has a structure of equation of the *eigenvalue problem* for differential operator ∇^2. It is called the ***Helmholtz Equation***.

The solution of the Helmholtz Equation subject to boundary conditions can be easily obtained by the eigenfunction expansion method.

Separation of variables – 2nd stage:

Assume $\Phi(x,y)=X(x)Y(y)$

Substitute into the Helmholtz Equation

$$\nabla^2(XY)\equiv X''Y+XY''=\zeta XY$$

Divide by XY

$$\frac{X''}{X}+\frac{Y''}{Y}=\beta$$

Separation of variables in the boundary conditions yield:

$$y\in(0,M) \quad \left[\Phi\right]_{x=0}=\left[X(0)\right]Y(y)=0 \quad \Rightarrow \quad \left[X\right]_{x=0}=0$$

$$y\in(0,M) \quad \left[\Phi\right]_{x=L}=\left[X(L)\right]Y(y)=0 \quad \Rightarrow \quad \left[X\right]_{x=L}=0$$

$$x\in(0,L) \quad \left[\Phi\right]_{y=0}=X(x)\left[Y(0)\right]=0 \quad \Rightarrow \quad \left[Y\right]_{y=0}=0$$

$$x\in(0,L) \quad \left[\Phi\right]_{y=M}=X(x)\left[Y(M)\right]=0 \quad \Rightarrow \quad \left[Y\right]_{y=M}=0$$

Note, that we have complete pairs of homogeneous boundary conditions both for X and Y.

Now, solve consequently the Sturm-Liouville problems for X and Y:

$$\frac{X''}{X} = -\frac{Y''}{Y} + \beta = \mu$$

Equation is separated. It yields first SLP:

$$\begin{array}{lcll} X'' - \mu X = 0 & & & \\ \left[X\right]_{x=0} = 0 & \overset{SLP}{\Rightarrow} & \mu = -\lambda_n^2 & n = 1,2,... \\ \left[X\right]_{x=L} = 0 & & X_n\left(x\right) & \end{array}$$

Then the second equation becomes:

$$-\frac{Y''}{Y} + \beta = -\lambda_n^2$$

which in its turn is a separated equation:

$$\frac{Y''}{Y} = \beta + \lambda_n^2 = \eta$$

It yields the second Sturm-Liouville Problem:

$$\begin{array}{lcll} Y'' - \eta Y = 0 & & & \\ \left[Y\right]_{y=0} = 0 & \overset{SLP}{\Rightarrow} & \eta = -\nu_m^2 & m = 1,2,... \\ \left[Y\right]_{y=M} = 0 & & Y_m\left(y\right) & \end{array}$$

Equation for separation constants yields:

$$\beta + \lambda_n^2 = -\nu_m^2 \qquad \Rightarrow \quad \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Then equation for T becomes

$$\frac{1}{\alpha}\frac{T'}{T} = \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Which is the 1st order ordinary differential equation:

$$T' + \alpha\left(\lambda_n^2 + \nu_m^2\right)T = 0$$

with the solutions:

$$T_{nm}\left(t\right) = e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Solution of the Transient Problem:

Construct the solution in the form of double infinite series (eigenfunction expansion):

$$U\left(x,y,t\right) = \sum_n \sum_m A_{nm} X_n Y_m e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Where the coefficients A_{nm} can be found from the initial condition

$$U\left(x,y,0\right) = U_0\left(x,y\right) = \sum_n \sum_m A_{nm} X_n Y_m$$

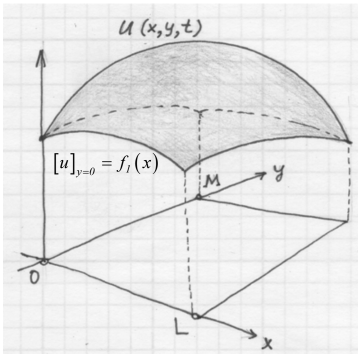
as the Fourier coefficients of the double Generalized Fourier series:

$$A_{nm} = \frac{\int_0^M \int_0^L U_0\left(x,y\right) X_n\left(x\right) Y_m\left(y\right) dx dy}{\left\|X_n\right\|^2 \left\|Y_m\right\|^2}$$

Example: DDNN

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{\alpha}\frac{\partial u}{\partial t}}$$

$u(x,y,t):\ (x,y)\in(0,L)\times(0,M)\ ,t>0$



Initial Condition: $u(x,y,0)=u_0(x,y)$

Boundary Conditions:

$x=0$	$[u]_{x=0}=0$	$y\in(0,M)$	$t>0$	(*Dirichlet*)
$x=L$	$[u]_{x=L}=f_4(y)$	$y\in(0,M)$	$t>0$	(*Dirichlet*)
$y=0$	$\left[\frac{\partial u}{\partial y}\right]_{y=0}=0$	$x\in(0,L)$	$t>0$	(*Neumann*)
$y=M$	$\left[\frac{\partial u}{\partial y}\right]_{y=M}=0$	$x\in(0,L)$	$t>0$	(*Neumann*)

---

1. Steady State Solution

Find time-independent solution $u_s(x,y)$:

$$\frac{\partial^2 u_s}{\partial x^2}+\frac{\partial^2 u_s}{\partial y^2}=0$$

subject to the boundary conditions:

$x=0$	$[u_s]_{x=0}=0$	$y\in(0,M)$	$t>0$
$x=L$	$[u_s]_{x=L}=f_4(y)$	$y\in(0,M)$	$t>0$
$y=0$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=0}=0$	$x\in(0,L)$	$t>0$
$y=M$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=M}=0$	$x\in(0,L)$	$t>0$

This is the basic problem for Laplace's Equation when, three boundary conditions are non-homogeneous.

Separation of variables: $u_s(x,y)=XY$

$x=0$	$[u_s]_{x=0}=0$	$\Rightarrow\ X(0)=0$
$x=L$	$[u_s]_{x=L}=f_4(y)$	
$y=0$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=0}=0$	$\Rightarrow\ Y'(0)=0$
$y=M$	$\left[\frac{\partial u_s}{\partial y}\right]_{y=M}=0$	$\Rightarrow\ Y'(M)=0$

Separated equation:

$$\frac{Y''}{Y}=-\frac{X''}{X}=\mu$$

First, consider equation for Y (two conditions):

$$Y''-\mu Y=0 \qquad \mu=-\lambda_n^2$$

$$Y'(0)=0 \quad \overset{SLP}{\Rightarrow} \quad \lambda_0=0 \qquad Y_0=1$$

$$Y'(M)=0 \qquad \lambda_n=\frac{n\pi}{M} \qquad Y_n(y)=\cos(\lambda_n y)=\cos\left(\frac{n\pi}{M}y\right)$$

Then equations for X :

$$X_0'' = 0 \qquad \Rightarrow \quad X_0(x) = c_1 + c_2 x$$

$$X_n'' - \lambda_n^2 X = 0 \qquad \Rightarrow \quad X_n(x) = c_1 \cosh(\lambda_n x) + c_2 \sinh(\lambda_n x)$$

Boundary condition at $x = 0$ yields

$$X_0(0) = 0 = c_1 + c_2 \cdot 0 = c_1 \qquad \Rightarrow \quad c_1 = 0$$

$$X_n(0) = 0 = c_1 \cdot 1 + c_2 \cdot 0 = c_1 \qquad \Rightarrow \quad c_1 = 0$$

Then

$$X_0(x) = x$$

$$X_n(x) = \sinh\left(\frac{n\pi}{M} x\right)$$

Construct the steady state solution as

$$u_s(x,y) = a_0 X_0 Y_0 + \sum_{n=1}^{\infty} a_n X_n Y_n \quad = a_0 x + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{M} x\right) \cos\left(\frac{n\pi}{M} y\right)$$

This solution should satisfy the boundary condition at $x = L$:

$$u_s(L,y) = f_3(y) = a_0 L + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{M} L\right) \cos\left(\frac{n\pi}{M} y\right)$$

Which is a cosine Fourier series expansion of $f_3(y)$ with

$$a_0 = \frac{1}{LM} \int_0^M f_3(y) dy$$

$$a_n = \frac{2}{M \sinh\left(\frac{n\pi}{M} L\right)} \int_0^M f_3(y) \cos\left(\frac{n\pi}{M} y\right) dy$$

Then the steady state solution becomes:

$$\boxed{u_s(x,y) = \left[\frac{1}{LM} \int_0^M f_3(y) dy\right] x + \sum_{n=1}^{\infty} \left[\frac{2}{M \sinh\left(\frac{n\pi}{M} L\right)} \int_0^M f_3(y) \cos\left(\frac{n\pi}{M} y\right) dy\right] \sinh\left(\frac{n\pi}{M} x\right) \cos\left(\frac{n\pi}{M} y\right)}$$

2. Transient Solution

Introduce the transient function as

$$U(x,y,t) = u(x,y,t) - u_s(x,y)$$

Function U satisfies *homogeneous* Heat Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{1}{\alpha} \frac{\partial U}{\partial t}$$

with four *homogeneous* boundary conditions:

$$x = 0 \qquad [U]_{x=0} = 0 \qquad y \in (0,M) \quad t > 0$$

$$x = L \qquad [U]_{x=L} = 0 \qquad y \in (0,M) \quad t > 0$$

$$y = 0 \qquad \left[\frac{\partial U}{\partial y}\right]_{y=0} = 0 \qquad x \in (0,L) \quad t > 0$$

$$y = M \qquad \left[\frac{\partial U}{\partial y}\right]_{y=M} = 0 \qquad x \in (0,L) \quad t > 0$$

and the initial condition:

$$U(x,y,0) = u_0(x,y) - u_s(x,y) \quad \equiv U_0(x,y)$$

Separation of variables $U = XYT$ yields a separated equation

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

with homogeneous boundary conditions:

$$x = 0 \quad [U]_{x=0} = 0 \quad \Rightarrow \quad X(0) = 0$$

$$x = L \quad [U]_{x=L} = 0 \quad \Rightarrow \quad X(L) = 0$$

$$y = 0 \quad \left[\frac{\partial U}{\partial y}\right]_{y=0} = 0 \quad \Rightarrow \quad Y'(0) = 0$$

$$y = M \quad \left[\frac{\partial U}{\partial y}\right]_{y=M} = 0 \quad \Rightarrow \quad Y'(M) = 0$$

Solve consequently the Sturm-Liouville problems for X and Y:

$$\frac{X''}{X} = -\frac{Y''}{Y} + \beta = \mu$$

$$\begin{array}{lcl} X'' - \mu X = 0 & & \\ [X]_{x=0} = 0 & \overset{DD}{\Longrightarrow} & \mu = -\lambda_n^2, \quad \lambda_n = \dfrac{n\pi}{L}, \quad n = 1,2,\ldots \\ [X]_{x=L} = 0 & & X_n(x) = \sin(\lambda_n x) = \sin\left(\dfrac{n\pi}{L}x\right) \end{array}$$

Then the second equation becomes:

$$-\frac{Y''}{Y} + \beta = \mu = -\lambda_n^2$$

which in its turn is a separated equation:

$$\frac{Y''}{Y} = \beta + \lambda_n^2 = \eta$$

It yields the second Sturm-Liouville Problem:

$$\begin{array}{lcll} Y'' - \eta Y = 0 & & \eta = -\nu_m^2 & m = 0,1,2,\ldots \\ [Y']_{y=0} = 0 & \overset{NN}{\Longrightarrow} & \nu_0 = 0 & Y_0(y) = 1 \\ [Y']_{y=M} = 0 & & \nu_m = \dfrac{m\pi}{M} & Y_m(y) = \cos\left(\dfrac{m\pi}{M}y\right) \end{array}$$

Equation for separation constants yields:

$$\beta + \lambda_n^2 = \eta = -\nu_m^2 \quad \Rightarrow \quad \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Then equation for T becomes

$$\frac{1}{\alpha}\frac{T'}{T} = \beta = -\left(\lambda_n^2 + \nu_m^2\right)$$

Which is the 1st order ordinary differential equation:

$$T' + \alpha\left(\lambda_n^2 + \nu_m^2\right)T = 0$$

with the solutions:

$$T_{nm}(t) = e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Solution of the Transient Problem:

Construct the solution in the form of double infinite series (eigenfunction expansion):

$$U(x,y,t) = \sum_{n=1}^{\infty} A_{n0} X_n Y_0 e^{-\alpha\lambda_n^2 t} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} X_n Y_m e^{-\alpha\left(\lambda_n^2 + \nu_m^2\right)t}$$

Where the coefficients A_{nm} can be found from the initial condition:

$$U(x,y,0)=U_0(x,y)=\sum_{n=1}^{\infty}A_{n0}X_nY_0+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}X_nY_m$$

$$U_0(x,y)=\left[\sum_{n=1}^{\infty}A_{n0}X_n\right]Y_0+\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}A_{nm}X_n\right]Y_m$$

where

$$\left[\sum_{n=1}^{\infty}A_{n0}X_n\right]=\frac{1}{M}\int_0^M U_0(x,y)dy$$

$$\left[\sum_{n=1}^{\infty}A_{nm}X_n\right]=\frac{2}{M}\int_0^M U_0(x,y)Y_m(y)dy$$

Then

$$A_{n0}=\frac{2}{LM}\int_0^L\int_0^M U_0(x,y)X_n(x)dydx$$

$$A_{nm}=\frac{4}{LM}\int_0^M\int_0^L U_0(x,y)X_n(x)Y_m(y)dxdy$$

3. Solution of IBVP

$$u(x,y,t)=U(x,y,t)+u_s(x,y)$$

$$u(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{M}y\right)e^{-\left(\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{M^2}\right)\frac{1}{a^2}t}$$

$$+a_0x+\sum_{m=1}^{\infty}a_m\sinh\left(\frac{m\pi}{M}x\right)\cos\left(\frac{m\pi}{M}y\right)$$

where coefficients are

$$A_{0n}=\frac{2}{LM}\int_0^L\int_0^M\left[g(x,y)-u_s(x,y)\right]\sin\left(\frac{n\pi}{L}x\right)dydx$$

$$A_{mn}=\frac{4}{LM}\int_0^L\int_0^M\left[g(x,y)-u_s(x,y)\right]\cos\left(\frac{m\pi}{M}y\right)\sin\left(\frac{n\pi}{L}y\right)dydx$$

$$a_0=\frac{1}{LM}\int_0^M f(y)dy$$

$$a_m=\frac{2}{M\sinh\left(\frac{m\pi}{M}L\right)}\int_0^M f(y)\cos\left(\frac{m\pi}{M}y\right)dy$$

4. Maple Example: *heat5dn-2.mws* $L=2$, $M=4$, $\alpha=0.5$, $f(y)=1$, $g(x,y)=x(x-L)+y(y-M)$

2-D Heat Equation Example DD-NN

```
> restart;
> with(plots):
> L:=2;M:=4;alpha:=0.5;
```

$$L := 2$$

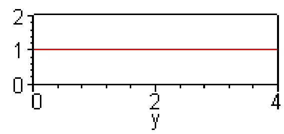
$$M := 4$$

$$\alpha := 0.5$$

```
> f(y):=1;
```

$$f(y) := 1$$

```
> plot(f(y),y=0..M,axes=boxed);
```



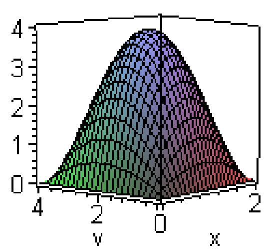
function in non-homogeneous boundary condition

$[u]_{x=L} = f_4(y)$

```
> u0(x,y):=x*(x-L)*y*(y-M);
```

$$u0(x,y) := x(x-2)\,y(y-4)$$

```
> plot3d(u0(x,y),x=0..L,y=0..M,axes=boxed);
```



initial temperature distribution

$u_0(x,y) = x(x-L)+y(y-M)$

Steady State Solution:

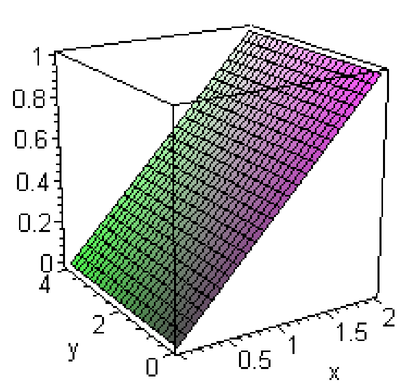
```
> a[0]:=int(f(y),y=0..M)/L/M;
```

$$a_0 := \frac{1}{2}$$

```
> a[m]:=2/M*int(f(y)*cos(m*Pi*y/M),y=0..M)/sinh(m*Pi*L/M);
```

$$a_m := \frac{2\sin(m\pi)}{m\pi\sinh\left(\frac{m\pi}{2}\right)}$$

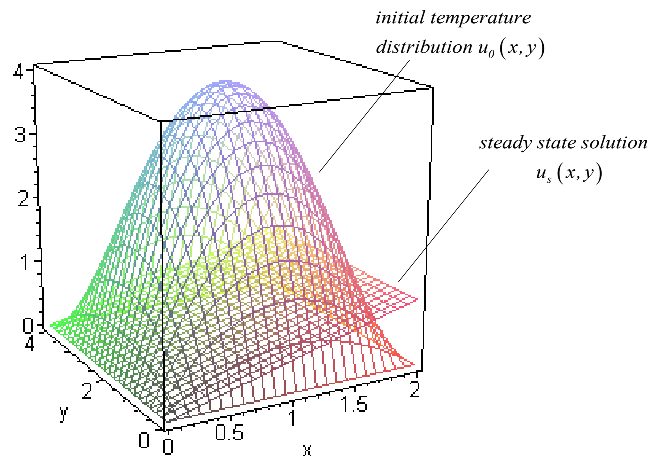
```
> us[m](x,y):=a[m]*sinh(m*Pi*x/M)*cos(m*Pi*y/M):
> us(x,y):=a[0]*x+sum(us[m](x,y),m=1..2):
> plot3d(us(x,y),x=0..L,y=0..M,axes=boxed,projection=0.92);
```



steady state solution

$u_s(x,y)$

```
> plot3d({us(x,y),u0(x,y)},x=0..L,y=0..M,axes=boxed,style=wireframe);
```



Transient Solution:

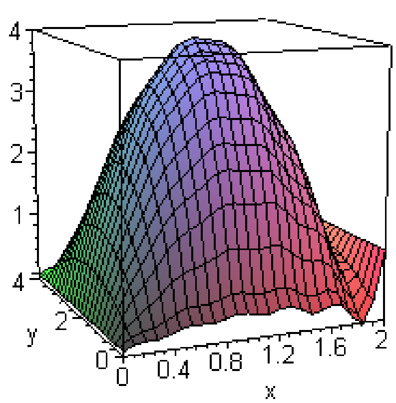
```
> U0(x,y):=u0(x,y)-us(x,y);
```

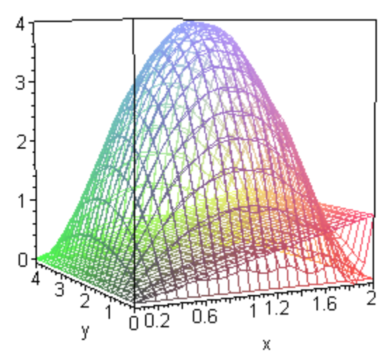
$$U0(x,y) := x\,(x-2)\,y\,(y-4) - \frac{x}{2}$$

```
> A[n,0]:=2*int(int(U0(x,y),y=0..M)*sin(n*Pi*x/L),x=0..L)/L/M:

> A[n,m]:=4*int(int(U0(x,y)*cos(m*Pi*y/M),y=0..M)*sin(n*Pi*x/L),x=0..L)/L/M:

> U[n,0](x,y,t):=A[n,0]*sin(n*Pi*x/L)*exp(-n^2/L^2*Pi^2*t*alpha):

> U[n,m](x,y,t):=A[n,m]*sin(n*Pi*x/L)*cos(m*Pi*y/M)*exp(-
  (m^2/M^2+n^2/L^2)*Pi^2*t*alpha):

>
U(x,y,t):=sum(U[n,0](x,y,t),n=1..10)+sum(sum(U[n,m](x,y,t),m=1..10),n=1..10):

> U(x,y,0):=subs(t=0,U(x,y,t)):
```

Solution of IBVP:

```
> u(x,y,t):=us(x,y)+U(x,y,t):

> u(x,y,0):=subs(t=0,u(x,y,t)):

> animate3d(u(x,y,t),x=0..L,y=0..M,t=0..3,frames=100,axes=boxed);
```

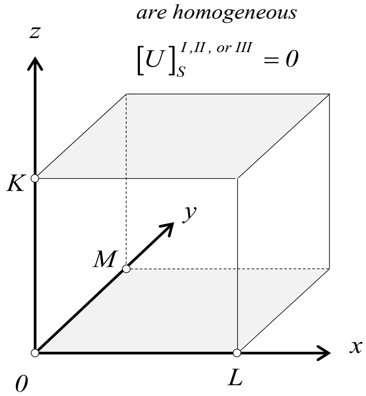




VIII.3.2.2 3-D TRANSIENT PROBLEM. HELMHOLTZ EQUATION.

all boundary conditions are homogeneous

$$\left[U\right]_S^{I, II, \text{ or } III} = 0$$



initial condition:

$$\left[U\right]_{t=0} = U_0\left(x,y,z\right)$$

Consider transient problem from the solution of the 3-D Heat Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad \left(x,y,z\right) \in \left(0,L\right)\times\left(0,M\right)\times\left(0,K\right) \;, t > 0$$

Separation of variables:

$$U\left(x,y,z,t\right) = \Phi\left(x,y,z\right)T\left(t\right)$$

Separated equation:

$$\frac{\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2} + \dfrac{\partial^2 \Phi}{\partial z^2}}{\Phi} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

Separated equation yields the Helmholtz Equation:

Helmholtz Equation

$$\boxed{\nabla^2 \Phi = \beta \Phi}$$

which constitutes the *eigenvalue problem* for differential operator ∇^2.

The solution of the Helmholtz Equation subject to boundary conditions can be easily obtained by the eigenfunction expansion method.

Assume

$$\Phi\left(x,y,z\right) = X\left(x\right)Y\left(y\right)Z\left(z\right)$$

Substitute into the Helmholtz Equation

$$\nabla^2\left(XYZ\right) \equiv X''YZ + XY''Z + XYZ'' = \beta XYZ$$

Divide by XYZ

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \beta$$

Separation of variables in the boundary conditions yields:

$$\begin{array}{llll}
x = 0 & \left[\Phi\right]_{x=0} = \left[X\left(0\right)\right]Y\left(y\right)Z\left(z\right) = 0 & \Rightarrow & \left[X\right]_{x=0} = 0 \\
x = L & \left[\Phi\right]_{x=L} = \left[X\left(L\right)\right]Y\left(y\right)Z\left(z\right) = 0 & \Rightarrow & \left[X\right]_{x=L} = 0 \\
y = 0 & \left[\Phi\right]_{y=0} = X\left(x\right)\left[Y\left(0\right)\right]Z\left(z\right) = 0 & \Rightarrow & \left[Y\right]_{y=0} = 0 \\
y = M & \left[\Phi\right]_{y=M} = X\left(x\right)\left[Y\left(M\right)\right]Z\left(z\right) = 0 & \Rightarrow & \left[Y\right]_{y=M} = 0 \\
z = 0 & \left[\Phi\right]_{z=0} = X\left(x\right)Y\left(y\right)\left[Z\left(0\right)\right] = 0 & \Rightarrow & \left[Z\right]_{z=0} = 0 \\
z = K & \left[\Phi\right]_{z=K} = X\left(x\right)Y\left(y\right)\left[Z\left(K\right)\right] = 0 & \Rightarrow & \left[Z\right]_{z=K} = 0
\end{array}$$

Note, that we have complete pairs of homogeneous boundary conditions for X, Y and Z.

Solve consequently the Sturm-Liouville problems for X, Y, and Z :

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} + \beta = \mu$$

Supplemental Eigenvalue problems

The first Sturm-Liouville Problem:

$$X'' - \mu X = 0$$
$$\left[X\right]_{x=0} = 0 \quad \overset{SLP}{\Longrightarrow} \quad \mu = -\lambda_n^2 \qquad n = (0), 1, 2, \ldots$$
$$\left[X\right]_{x=L} = 0 \qquad X_n(x)$$

Then the equation becomes:

$$-\frac{Y''}{Y} - \frac{Z''}{Z} + \beta = \mu = -\lambda_n^2$$

which in its turn is a separated equation:

$$\frac{Y''}{Y} = -\frac{Z''}{Z} + \beta + \lambda_n^2 = \eta$$

It yields the second Sturm-Liouville Problem:

$$Y'' - \eta Y = 0$$
$$\left[Y\right]_{y=0} = 0 \quad \overset{SLP}{\Longrightarrow} \quad \eta = -\nu_m^2 \qquad m = (0), 1, 2, \ldots$$
$$\left[Y\right]_{y=M} = 0 \qquad Y_m(y)$$

Then one more step produces equation

$$-\frac{Z''}{Z} + \beta + \lambda_n^2 = -\nu_m^2$$

which also can be separated

$$\frac{Z''}{Z} = \beta + \lambda_n^2 + \nu_m^2 = \gamma$$

It yields the third Sturm-Liouville Problem:

$$Z'' - \gamma Z = 0$$
$$\left[Z\right]_{z=0} = 0 \quad \overset{SLP}{\Longrightarrow} \quad \gamma = -\omega_k^2 \qquad k = (0), 1, 2, \ldots$$
$$\left[Z\right]_{z=K} = 0 \qquad Z_k(z)$$

Then the second part of the last equation becomes

$$\beta + \lambda_n^2 + \nu_m^2 = -\omega_k^2$$

and the constant of separation is

$$\beta_{nmk} = -\left(\lambda_n^2 + \nu_m^2 + \omega_k^2\right)$$

Then the solution of the Basic IBVP for the Heat Equation is:

Solution of Basic IBVP:

$$U(x,y,z,t) = \sum_n \sum_m \sum_k B_{nmk} X_n(x) Y_m(y) Z_k(z) e^{-\alpha\left(\lambda_n^2 + \nu_m^2 + \omega_k^2\right)t}$$

where the coefficients B_{nmk} can be found from the initial condition as the Fourier coefficients of the triple Generalized Fourier Series:

$$B_{nmk} = \frac{\int_0^K \int_0^M \int_0^L U_0(x,y,z) X_n(x) Y_m(y) Z_k(z)\, dx dy dz}{\left\|X_n\right\|^2 \left\|Y_m\right\|^2 \left\|Z_k\right\|^2}$$

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